Solution Set 8

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1 Problem 1

From Electrodynamics, we know that the continuity equation is $\vec{\nabla} \cdot \vec{J} = -\frac{d\rho}{dt}$. In one dimension this reduces to

$$\frac{dj}{dx} = -\frac{d\rho}{dt}$$

and

$$\frac{dj}{dx} = \frac{-\hbar}{2mi}\left(\frac{d^2\psi^*}{dx^2}\psi - \frac{d^2\psi}{dx^2}\psi^*\right).$$

With the use of the SE we also get

$$\frac{d\rho}{dt} = \frac{d\psi^*}{dt}\psi + \psi^*\frac{d\psi}{dt} = \frac{-\psi}{i\hbar}\left(\frac{-\hbar^2}{2m}\frac{d^2\psi^*}{dx^2} + U^*(x)\psi^*\right) + \frac{\psi^*}{i\hbar}\left(\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi\right) = -\frac{dj}{dx} + \frac{(U - U^*)\psi^*\psi}{i\hbar}$$

Hence the continuity equation is not satisfied when U^* differs from U.

2 Problem 2

If the coordinate system is kept as given, then it is necessary to solve the SE for it with the new boundary conditions

$$\psi(-L/2) = \psi(L/2) = 0.$$

If you try the solution of the form $\psi(x) = A\cos(kx) + B\sin(kx)$ then because you no longer have the condition $\psi(0) = 0$, both sines and cosines will be legitimate solutions. Thus, the simpliest thing to do is to shift coordinates from x to y, where $y = x + L/2$. Then the potential extends from $0 < y < L$ ($\psi(y = 0) = 0$) and

$$\psi_n(y) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi y}{L}\right).$$

a). Then for the particle in the $n'th$ eigenstate the probability is

$$\int_0^{L/2} \psi_n^*(y)\psi_n(y)dy = 2/L\int_0^{L/2} \sin^2\left(\frac{n\pi y}{L}\right)dy = 2/L\int_0^{L/2} \frac{1}{2}\left(1 - \cos\left(\frac{2n\pi y}{L}\right)\right)dy = 1/2,$$

where the integral of the cosine vanished because of the pereodicity and the probability is independent of n.

b). Similarly this probability is

$$\int_{L/2}^{3L/4} \psi_n^*(y)\psi_n(y)dy = 2/L \int_{L/2}^{3L/4} \sin^2(\frac{n\pi y}{L})dy = 2/L \int_{L/2}^{3L/4} \frac{1}{2}(1-\cos(\frac{2n\pi y}{L}))dy$$
$$= 1/2 - \frac{1}{2n\pi}\left(\sin(3n\pi/2) - \sin(n\pi/2)\right)$$

For the ground state this quantity is $1/2 + 1/\pi$ and for large n it is $1/2$.

3 Problem 3

In order to solve this problem I will use the orthogonality of eigenstates, i.e.

$$\int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x)dx = \delta_{mn},$$

where ψ_n is the properly normalized $n'th$ eigenstate, and δ_{mn} is 1 for $m = n$ and 0 otherwise.

a). Thus

$$u(x) = c_1\psi_1 + c_2\psi_2$$

with $c_2 = 2c_1$. Then

$$1 = \int_{-\infty}^{\infty} u^*(x)u(x)dx = \int_{-\infty}^{\infty} (|c_1|^2|\psi_1|^2 + c_1c_2^*\psi_1\psi_2^* + c_1^*c_2\psi_1^*\psi_2 + |c_2|^2|\psi_2|^2)dx = |c_1|^2 + |c_2|^2,$$

where orthogonality was used in the last step. Thus (up to a phase) $c_1 = 1/\sqrt{5}$ and $c_2 = 2/\sqrt{5}$.

b). The operator E acting on the $n'th$ eigenstate gives E_n (i.e. $E\psi_n = E_n\psi_n$). Thus

$$\begin{aligned} < E > &= \int_{-\infty}^{\infty} u^*(x)Eu(x)dx = \int_{-\infty}^{\infty} (E_1|c_1|^2|\psi_1|^2 + E_1c_1c_2^*\psi_1\psi_2^* + E_2c_1^*c_2\psi_1^*\psi_2 + E_2|c_2|^2|\psi_2|^2)dx \\ &= E_1|c_1|^2 + E_2|c_2|^2. \end{aligned}$$

Using the fact that for 1-D well $E_n = \frac{\hbar^2\pi^2}{2mL}n^2$,

$$< E >= \frac{\hbar^2\pi^2}{2mL}(9/5).$$

c). When the measurement is taken the particle will be found in either first or second eigenstate (i.e. we will measure either E_1 or E_2). In general, the probability of finding a particle in the $n'th$ eigenstate is

$$P_n = |\int_{-\infty}^{\infty} \psi_n^*(x)u(x)dx|^2.$$

In our case

$$P_1 = |\int_{-\infty}^{\infty} \psi_1^*(x)u(x)dx|^2 = |\int_{-\infty}^{\infty}(c_1|\psi_1|^2 + c_2\psi_1\psi_2^*)dx|^2 = |c_1|^2 = 1/5$$

and similarly

$$P_2 = |\int_{-\infty}^{\infty} \psi_2^*(x)u(x)dx|^2 = |c_2|^2 = 4/5.$$

From statistics we know that the expectation value of a quantity is the sum over possible values times the probability of each value; i.e.

$$< E >= \sum_n E_n P_n = E_1|c_1|^2 + E_2|c_2|^2,$$

which is the same as in part b).

d). The measurement of energy will collapse the wavefunction down to a single state, and $< E >$ will equal the value measured. Thus $< E >$ will be E_1 or E_2.

4 Problem 4

a).

$$\begin{aligned} < x > &= \int_{-\infty}^{\infty} u^*(x)xu(x)dx = (2/L)|c_1|^2 \int_0^L x\left(\sin(x\pi/L) + 2\sin(2x\pi/L)\right)^2 dx \\ &= 2L|c_1|^2 \int_0^1 y\left(\sin(y\pi) + 2\sin(2y\pi)\right)^2 dy, \end{aligned}$$

where I used substitution $y = x/L$ in the last step, in order to eliminate L dependence of the integral. It is now purely numerical and can easily be evaluated on a calculator (or you can do it by parts) and $< x >= 0.7L|c_1|^2 = 0.14L$.

b). The time dependence is added by multiplying each eigenstate by $e^{-i/\hbar E_n t}$; i.e. for $\psi(x,0) = c_1\psi_1 + c_2\psi_2$

$$\psi(x,t) = c_1\psi_1 e^{-i/\hbar E_1 t} + c_2\psi_2 e^{-i/\hbar E_2 t}$$

c). Now

$$\begin{aligned} < x > &= \int_{-\infty}^{\infty} u^*(x)xu(x)dx = (2/L)|c_1|^2 \int_0^L x\left(e^{-i/\hbar E_1 t}\sin(x\pi/L) + 2e^{-i/\hbar E_2 t}\sin(2x\pi/L)\right)^2 dx \\ &= 2L|c_1|^2 \int_0^1 y\left(\sin(y\pi) + 2e^{-i\theta t}\sin(2y\pi)\right)^2 dy \\ &= 2L|c_1|^2 \int_0^1 y\left(\sin^2(y\pi) + 4\sin^2(2y\pi) + 2(e^{i\theta t} + e^{-i\theta t})\sin(y\pi)\sin(2y\pi)\right) dy, \end{aligned}$$

where $\theta = (E_2 - E_1)/\hbar$ and

$$2L|c_1|^2 \int_0^1 y2(e^{i\theta t} + e^{-i\theta t})\sin(y\pi)\sin(2y\pi)dy = 8L\cos(t\theta)|c_1|^2 int_0^1 y\sin(y\pi)\sin(2y\pi)dy < 0$$

The last integral is less than 0 because $\sin(y\pi)\sin(2y\pi)$ is antisymmetric about $y = 0.5$ and negative for $y > 0.5$ (and clearly y takes on greater values for $y > 0.5$). Thus $< x >$ will be the smallest whenever $\cos(\theta t) = 1$, which happends at $t = 0$. Similarly $< x >$ will be the greatest whenever $\cos(\theta t) = 0$; this first happends for

$$t = \pi/(2\theta) = \frac{\pi\hbar}{2(E_2 - E_1)} = \frac{\pi\hbar}{6E_1}$$

5 Problem 5

a).

$$\Delta p \Delta x \simeq \hbar/2$$

The momentum roughly equals it's uncertainty, thus

$$E = \frac{p^2}{2m} \simeq \frac{\hbar^2}{8m\Delta x^2}.$$

Since the uncertainty in position is the diameter of the proton ($4fm$), $E = 9.5 * 10^{-11}J = 6 * 10^3 MeV$.

b). For a classical oscillator $F = kx$ and the Kinetic energy at the center equals potential energy at maximum displacement, i.e. $1/2kx^2 = E$ and

$$F = 2E/x = 6 * 10^3 MeV/fm$$

c). In this case

$$F = \frac{e^2}{4\pi\epsilon_0 x^2} = \frac{\alpha\hbar c}{x^2} = 0.36 Mev/fm,$$

which is much less than the answer to part b) (thus the proton is confined to the nucleus).

6 Problem 6

In two dimensions the time independent SE is

$$\frac{-\hbar^2}{2m}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + U(x,y)u = Eu.$$

Once again $U = 0$ in the well and $u(x,y) = 0$ outside, thus we have the boundary conditions

$$u(0,y) = u(x,0) = u(L,y) = u(x,L) = 0.$$

Since u must be zero whenever x or y is 0 the solutions are of the form

$$u(x,y) = A\sin(k_x x)\sin(k_y y).$$

Since $u(L,y) = u(x,L) = 0$ we have the following restrictions on k_x and k_y

$$k_x L = n_x \pi$$

and

$$k_y L = n_y \pi,$$

where $n_x, n_y = 1, 2, ...$ Normalizability now states that

$$1 = \int_0^L dx \int_0^L dy A^2 \sin^2(n_x \pi x/L) \sin^2(n_y \pi y/L),$$

and therefore $A = 2/L$. Hence we have worked out the eigenstates to be

$$u_{n_x,n_y}(x,y) = \frac{2}{L} \sin(n_x \pi x/L) \sin(n_y \pi y/L).$$

Plugging these back into SE gives us the corresponding energies

$$E_{n_x,n_y} = \frac{\hbar^2 \pi^2}{2mL}(n_x^2 + n_y^2).$$

7 Problem 7

Near the ground state $\Delta p \simeq p$. Thus (using $\Delta p \Delta x \simeq \hbar/2$)

$$E = p^2/(2m) + U \simeq \frac{\hbar^2}{8m\Delta x^2} + \lambda \Delta x^4,$$

and we find the minimum of E by differentiating with respect to Δx, i.e.

$$0 = dE/d\Delta x = \frac{-\hbar^2}{4m\Delta x^3} + 4\lambda \Delta x^3.$$

Solving for Δx and substituting back into E gives us

$$E = \frac{\hbar^{4/3} 4^{2/3} \lambda^{1/3}}{8m^{2/3}} + \frac{\lambda^{1/3} \hbar^{4/3}}{4^{1/3} m^{2/3}} = \frac{\lambda^{1/3} \hbar^{4/3}}{m^{2/3}}(3/2^{5/3})$$

and the units work out correctly.

8 Problem 8

Since the virtual particle is created as a result of uncertainty in energy, it's rest energy $(140MeV)$ is roughly ΔE. Thus

$$d = vt \simeq c\hbar/(\Delta E) = 1.4fm.$$